Structured products: perfect solution or source of confusion?

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Overview

- Introduction
- Structured Products and Equity Indexed Annuities
- Current landscape
- Our perspective
- Example
- What sort of contract does Jane Doe want?
- Proposals for an optimal design.
- Basic result
- Focus on the intuition
- Examples
- Testing the design feasibility

Motivation

- Problem is of intellectual interest
- Lies at interface of finance and insurance
- Is also of practical interest
- These contracts are big business

What we do not consider

- Modeling of investment returns
- Econometrics
- Pricing
- Hedging
- Reserving (actuarial viewpoint)
- Risk management

Current Scene

- Bewildering range of investment products for retail investors
- These products often have following features
  1. Tied to performance of an equity index
  2. Downside protection in bear markets
  3. Participate in upside appreciation in bull markets.
  4. Can have complicated structure
  5. Contain embedded options
- Banks market structured products
- Insurance companies sell *equity indexed annuities(USA), segregated funds(Canada), equity linked contracts(Europe and Asia)*

Example : Guaranteed Equity Bond (UK)

Observer April 30 2006

Firm X is selling a guaranteed equity bond that will return 112 percent of growth in the FTSE 100 index over its five year term. The bond which goes on sale Wednesday guarantees to return all of the investor's capital if the index falls over this period.

Company Website

In todays market, an investment opportunity like this is too good to miss. But you have to act fast. Our GEB Issue 11 is only available for a limited period the offer closes at 6pm on 27 June 2006 or earlier if fully subscribed

Equity Indexed Annuity: USA

- Policyholder invests x_0 at time zero
- Contract matures at time T
- Ignore all frictions: expenses, transaction costs and mortality.
- Payoff at time T based on performance of reference index (eg S &P 500). Normally based on the capital value only. *No dividends*
- If S_t is index value at time t, payoff at maturity is equal to

$$\max \left[x_0 H \left(\frac{S_T}{S_0} \right), \; x_0 e^{\gamma T} \right]$$

where γ is the minimum guaranteed rate and H is the payoff factor. Investors love guarantees

Example: Point to point design

- Popular design in USA.
- Called *point to point* method because payoff is based on the value of the index at two (time) points.
- Contract has a participation rate equal to k.
- May also have interest rate cap c which limits the upside growth.
- Payoff at time T is given by

$$x_0 \ \min\left[\ \max\left(\left(\frac{S_T}{S_0}\right)^k, \ e^{\gamma T}\right), \ e^{cT} \ \right] \tag{1}$$

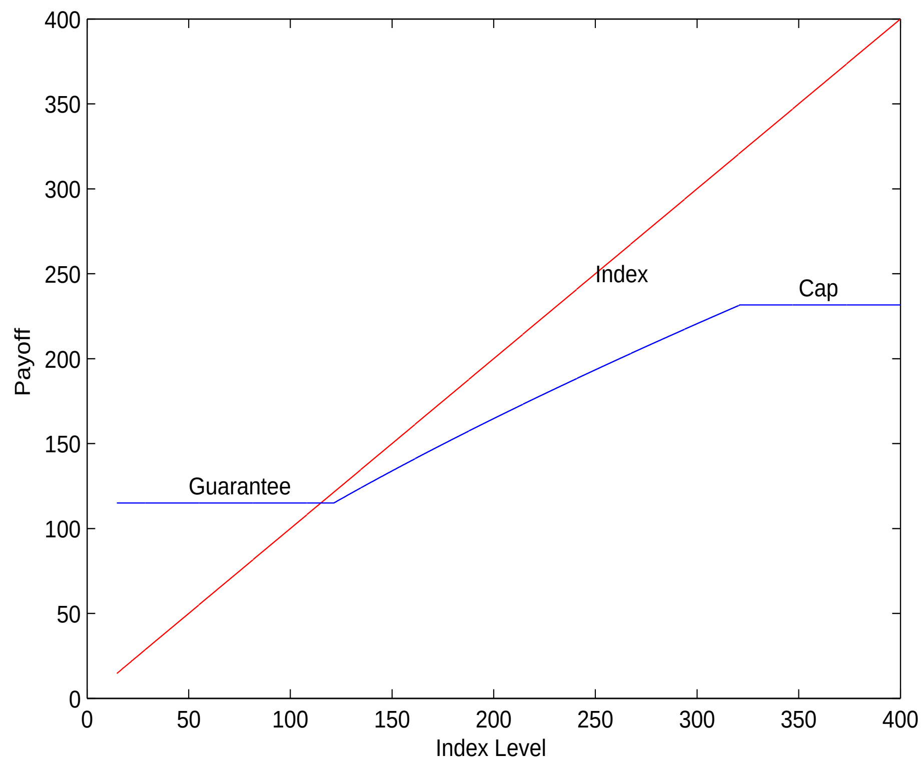

Figure 1: Blue line: payoff on equity indexed annuity contract. Guarantee $\gamma = .02$ pa. Participation rate 80% of index. Cap is 12%. Red line is payoff if fully invested in index. T =7, $\sigma = .2$, r=.05

Parameter restrictions

- Can obtain an exact expression for payoff when index is lognormal.
- No arbitrage gives relationship among the policy parameters $\{k, \gamma, c\}$

$$e^{rT} \geq f\Phi(\alpha) + e^{cT}\Phi(-\beta) + e^{kT(r+\frac{k-1}{2}\sigma^2)}\{\Phi(\beta - k\sigma\sqrt{T}) - \Phi(\alpha - k\sigma\sqrt{T})\}$$

where r is the risk-free rate $\Phi()$ is the cumulative normal distribution function,

$$\alpha = \frac{\gamma - k(r - \frac{1}{2}\sigma^2)}{k\sigma}\sqrt{T}, \quad \beta = \frac{c - k(r - \frac{1}{2}\sigma^2)}{k\sigma}\sqrt{T}$$

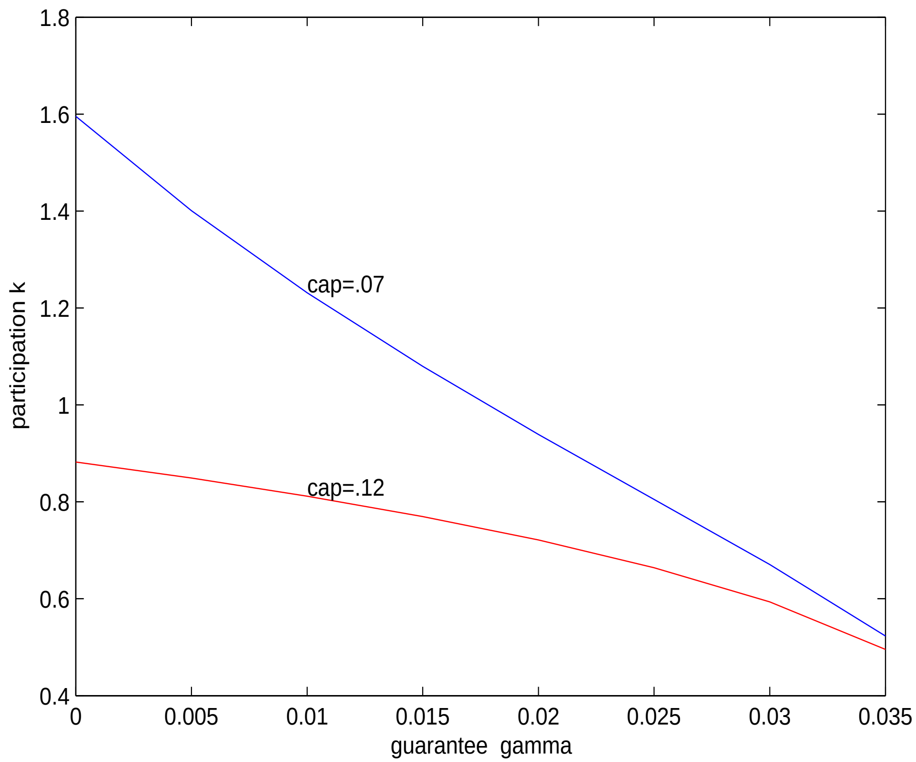

Figure 2: Relation between k and γ for different c. Blue line: participation rate on equity indexed annuity contract for cap=7%. Red line participation rate on equity indexed annuity contract for cap=12%

Merton's Model of Portfolio Selection

Robert Merton solved some of the fundamental problems in the area.

- Merton considered the optimal strategy of investor
- Available assets: the market and the risk free bond
- She rebalances her portfolio continuously
- Objective is to maximize expected utility
- Merton derived the solution under certain assumptions
- Lognormal returns and CRRA utility

The Merton solution

- Investor will follow a very simple strategy.
- Position in market will be a *constant fraction of her current wealth.*
- The constant proportion, often called the Merton ratio, is

$$\frac{Equity\ risk\ premium}{(Variance\ of\ market)\ (relative\ risk\ aversion)}$$

- Nice intuition

The Merton Ratio

Suppose

- Equity risk premium is 3%.
- Volatility of the market is 15%,
- relative risk aversion is 2

Merton ratio in this case is

$$\frac{.03}{2(0.15^2)} = 66.67\%$$

Hence 66.67% of her wealth should be invested in the market

Close to the 60-40 equity bond *rule of thumb* advocated by some investment pundits.

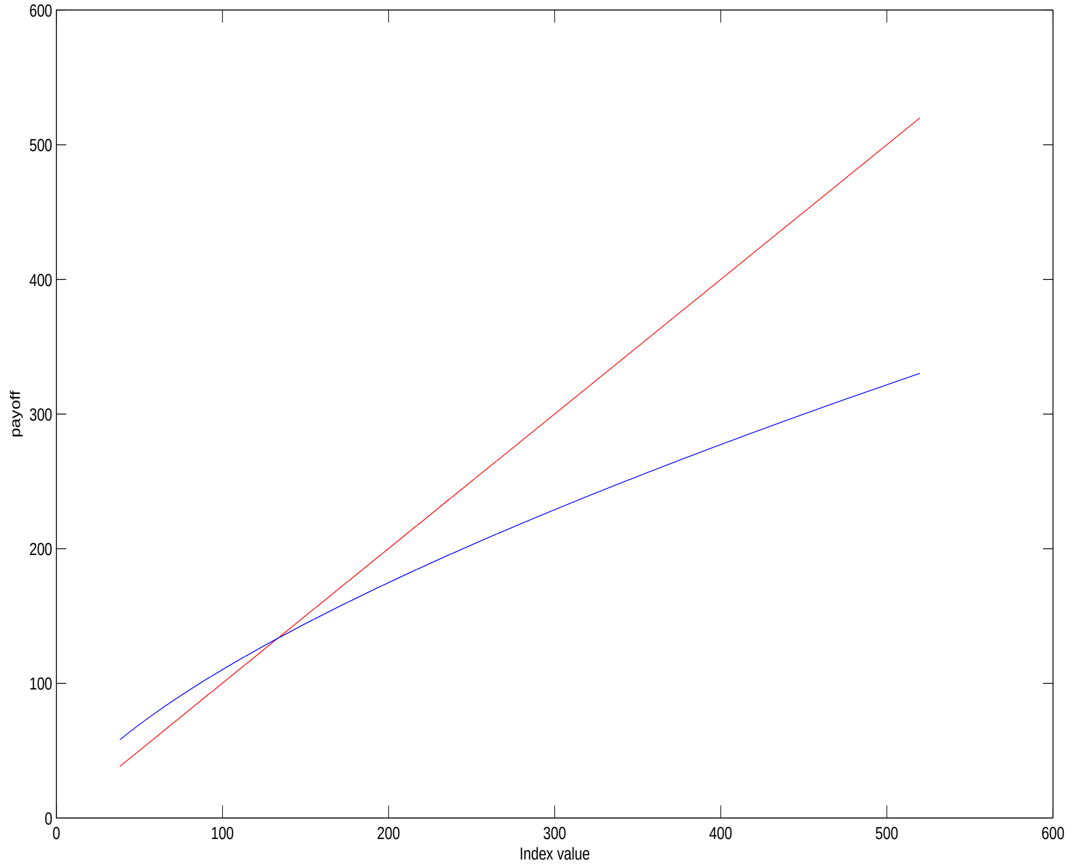

Figure 3: Optimal portfolio as function of market index under Merton model. Index is portfolio on red line. Investor with risk aversion of 2 prefers the blue line.

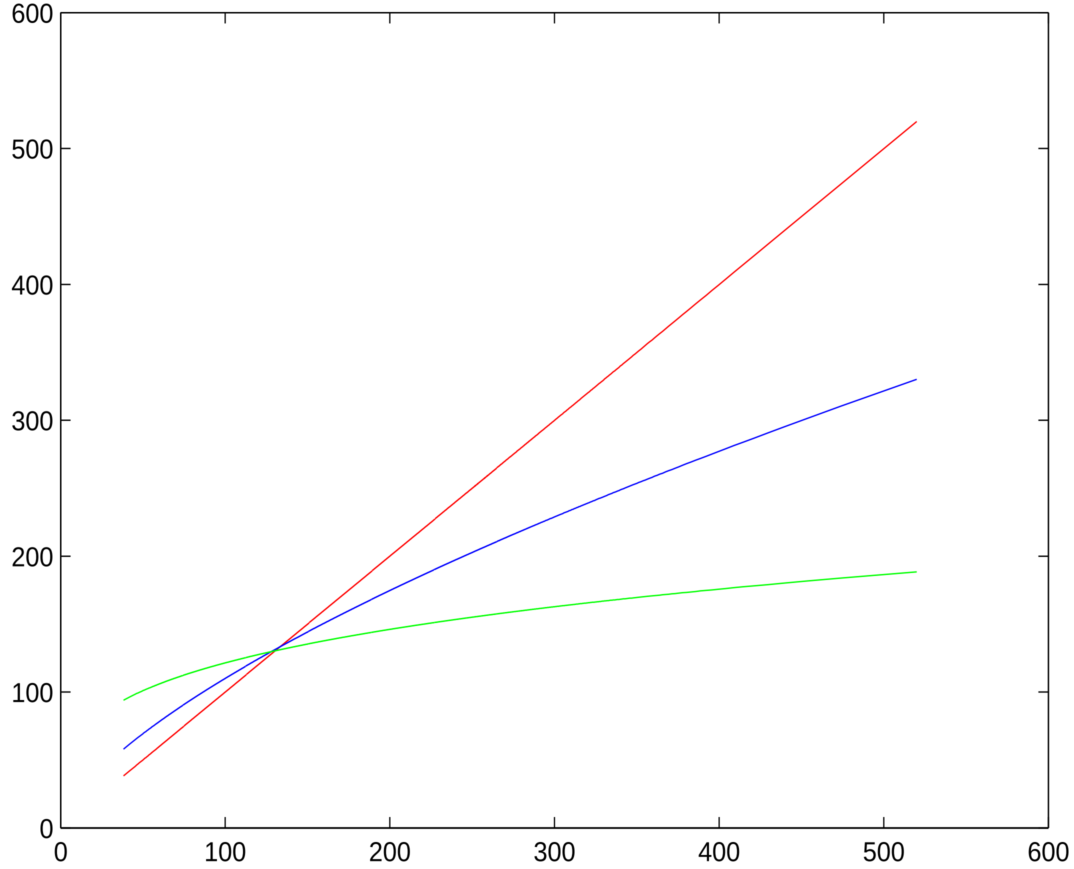

Figure 4: Optimal portfolio as function of market index under Merton model. Index is portfolio on red line. Blue line risk aversion is 2. Green line risk aversion is 5

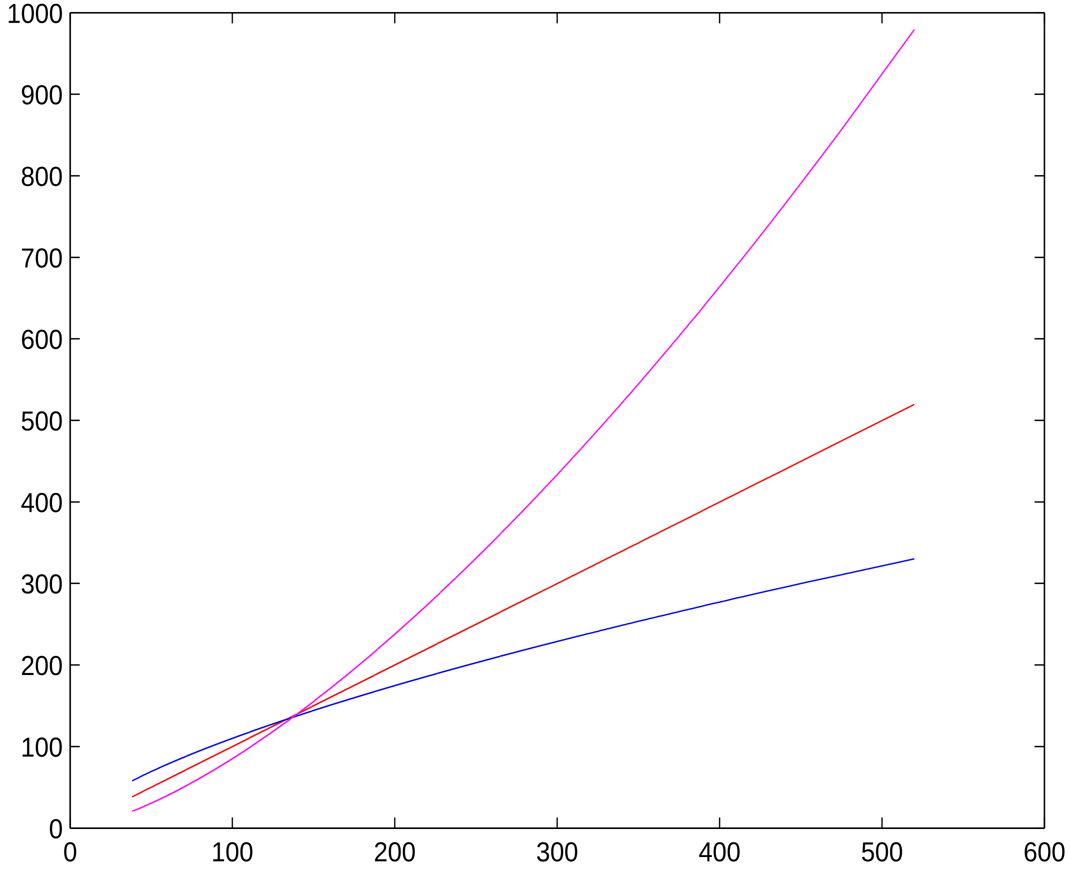

Figure 5: Optimal portfolio as function of market index under Merton model. Index is portfolio on red line. Blue line risk aversion is 2. Magenta line risk aversion is 0.9

Is current design of EIA optimal?

- Probably not from the investor's viewpoint
- But can we design a better contract?
- Our approach

  1. We propose *desirable* contract features

  2. Find a contract that has these features

  3. Basic idea Cox Huang Pliska martingale approach

  4. Draw on recent work by Boyle and Tian(2006)

  5. Here we do not emphasize the technical details

  6. Just describe the solution

Features of good design

Here are our proposals for a good design

1. Contract should maximize investor's expected utility.

2. Should include a minimum guarantee

3. We will give investor the opportunity to beat a benchmark with probability α. Common desire to beat benchmark. Cannot do it for sure

4. No rip off: must satisfy the no arbitrage condition

We assume complete markets and no arbitrage.

Martingale Approach

- Cox Huang, Karatzas, Lehoczky and Shreve and Pliska
- Two steps.
  1. First obtain the investor's optimal wealth.
  2. Then find portfolio that replicates this wealth
- Investor's final wealth can be viewed as a contingent claim which can be replicated in a complete market.
- *Often* easier to include constraints when using the martingale method
- The probabilistic constraint is *non convex* which can be *hard* see Basak and Shapiro(2001) for a VaR example

Tyrrell Rockafeller(1993)

..in fact, the great watershed in optimization isn't between linearity and non linearity but between convexity and non convexity

No constraints first: Merton problem

- Let $\{\xi_T : T > 0\}$ be the state-price density process
- Assume x_0 is the initial investment.
- Investor's preferences are represented by a utility function, $u(.)$
- the optimal portfolio selection problem is to solve

$$\max_{\pi} \; E[u(X_T^\pi)]$$

  for $\{\pi\}$ across all adapted trading strategies.

- Let $I(x) = (u')^{-1}(x)$ be the inverse of the first derivative of u

Cox Huang solution

- There exists an admissible process π_t^* with terminal wealth $X^{x_0,\pi^*}(T)$ such that

$$E[u(X^{x_0,\pi^*}(T))] \geq E[u(X^{x_0,\pi}(T))]$$

  for any admissible process $\{\pi_t\}$.

- $X^u(T) := X^{x_0,\pi^*}(T)$ is the *optimal terminal wealth.* $X^u(T) = I(\lambda\xi_T)$ for some positive λ.

- Select λ to satisfy the budget constraint; $E[\xi_T X^u(T)] = x_0$.

- The admissible process is *unique* in the sense that if there is another optimal terminal wealth $X^{x_0,\pi}(T)$, then $X^{x_0,\pi}(T) = X^u(T), a.s.$

Idea of the solution with constraints

To derive the optimal contract we use a constructive proof. Here are the steps.

- We postulate the functional form of a family of random variables indexed by a parameter λ.
- Then we show there exists one member of this class with $\lambda = \lambda^*$. which corresponds to the optimal terminal wealth for our desired contract

To ensure existence we need several technical conditions. These can be verified in any application. Many of them are related to the continuity of certain functions.

Constraints for the optimal EIA

The constraints are

- Guaranteed return: We require

$$X(T) \geq x_0 e^{\gamma T}$$

- Beating the benchmark: Suppose the benchmark is $\Gamma > 0$. Constraint is

$$P(X(T) \geq \Gamma) \geq \alpha$$

- No rip off

$$E[\xi_T X(T)] = x_0$$

Constructing the sequence

We now introduce a sequence of positive random variablesindexed byλ

$$\{X_{\lambda,\alpha}(T) : \lambda > \lambda_\alpha\}$$

as follows:

$$X_{\lambda,\alpha}(T) = \begin{cases} I(\lambda\xi_T), & if I(\lambda\xi_T) \geq \Gamma > f x_0 \\ \max\{I(\lambda\xi_T), f x_0\}, & if \ \max\{I(\lambda\xi_T), f x_0\} < \Gamma, h(\lambda, \xi_T, \alpha) > 0 \\ \Gamma, & if \max\{I(\lambda\xi_T), f x_0\} < \Gamma, h(\lambda, \xi_T, \alpha) \leq 0 \\ max\{I(\lambda\xi_T), f x_0\}, & if \ \Gamma \leq f x_0 \end{cases}.$$

where $f = e^{\gamma T}$ and λ_α will be defined shortly. Construction organized so that

$$P(\ max\{I(\lambda\xi_T), f x_0\} < \Gamma, h(\lambda, \xi_T, \alpha) > 0) = 1 - \alpha$$

Definition of λ_α

- Define

$$H(\lambda) = P(\max\{I(\lambda\xi_T), fx_0\} < \Gamma), \;\; \lambda > 0$$

and let

$$\lambda_\alpha := Sup\{\lambda : H(\lambda) < 1 - \alpha\}$$

.

- Under some technical assumptions concerning continuity $H(\lambda_\alpha) = 1 - \alpha$

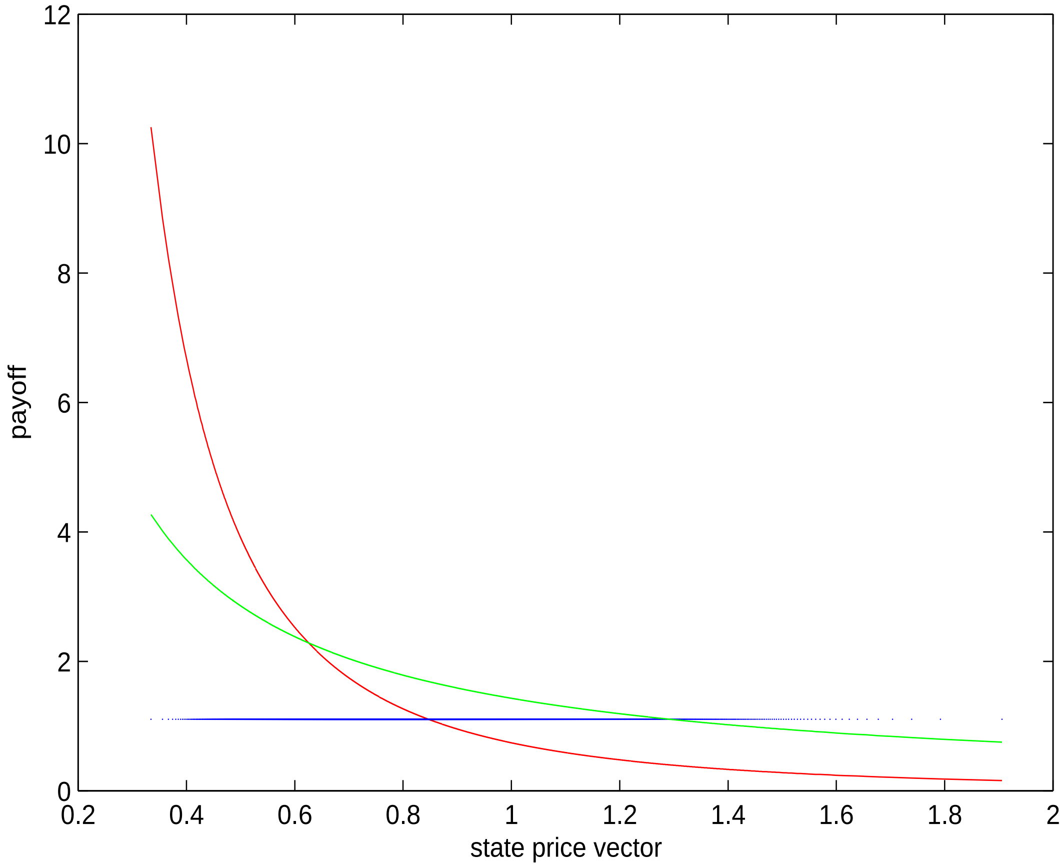

Figure 6: Construction of λ_α. Γ is red line. Green line is $I(\lambda\xi_T)$. Blue line is guaranteed amount. X-axis state price vector. Parameters $\mu = .06, r = .04, \sigma = .2, k = 1.2, g = .02, \alpha = .8625$ **,** $H(\lambda_\alpha) = .1375$

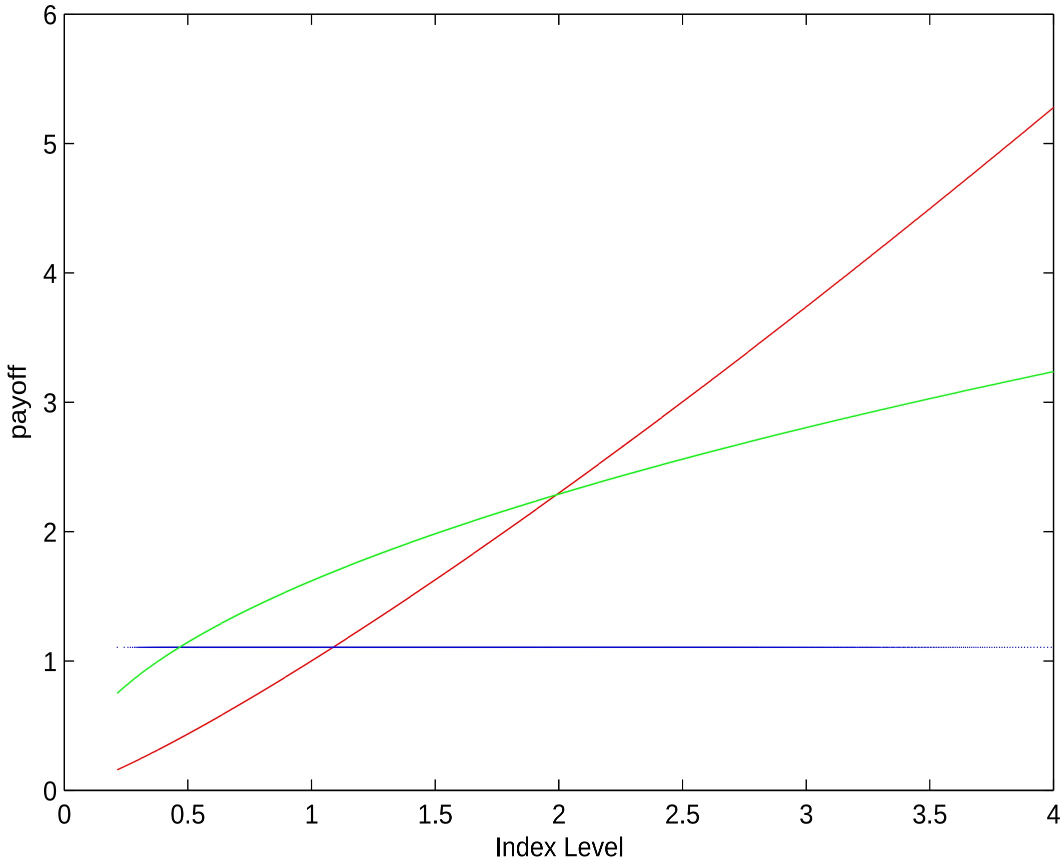

Figure 7: Construction of λ_α. Γ is red line. Green line is $I(\lambda\xi_T)$. Blue line is guaranteed amount. X-axis index price level. Parameters $\mu = .06, r = .04, \sigma = .2, k = 1.2, g = .02, \alpha = .8625$, $H(\lambda_\alpha) = .1375$

Main result

Under certain technical assumptions and assuming that

$$\overline{lim}_{\lambda\to\infty}\{E[\xi_T f x_0 1_{\{\Gamma\le f x_0\}}] + E[\xi_T f x_0 1_{\{\Gamma > f x_0,\ h(\lambda,\xi_T,\lambda,\alpha))\ge\ 0\}}]$$
$$+\ E[\xi_T \Gamma 1_{\{\Gamma > f x_0,\ h(\lambda,\xi_T,\lambda,\alpha)<\ 0\}}]\} < x_0,$$

then there exists an adapted process π_t^* with terminal wealth $X^e(T)$ such that

$$X^e(T) \ge f x_0,\ P(X^e(T)\ge\Gamma)\ge\alpha$$

Furthermore $E[u(X^e(T))] \ge E[u(X^{x_0,\pi}(T))]$ for any adapted process π_t whose terminal wealth subject to the constraint conditions:

$$X^{x_0,\pi}(T)\ge f x_0, P(X^{x_0,\pi}(T)\ge\Gamma)\ge\alpha$$

Main result continued

Moreover, we can choose

$$X^e(T) = X_{\lambda^*,\alpha}(T)$$

for some positive real number $\lambda^* > \lambda_\alpha$.

This result gives an explicit construction for the optimal EIA. Once we know the optimal terminal wealth it can be replicated .

Example of Optimal EIA

- Assume that
$$\Gamma = x_0 \left(\frac{S_T}{S_0}\right)^k$$
- Assume guaranteed rate is γ so $f = e^{\gamma T}$
- Assume $u(x) = log(x)$
- Assume that index is lognormal with drift μ and volatility σ
- Let
$$b = \frac{\mu - r}{\sigma^2}$$
- The form of the solution differs depending on whether

$$k < b,\ k = b,\ k > b$$

Base case Parameters

Assume

Parameter	Value
T	5
k	.75
g	0.02
r	.04
μ	0.06
σ	.20
α	.85
b	.5

so $k > b$.

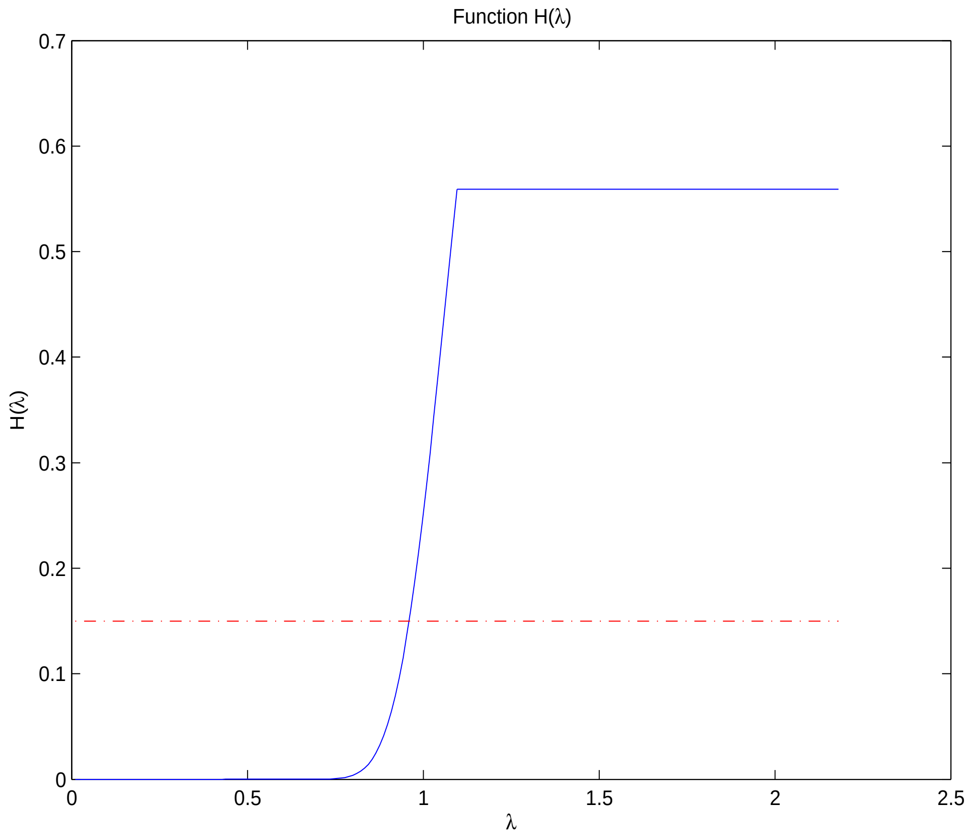

Figure 8: Graph of $H(\lambda)$ versus λ. Parameters $\mu = .06, r = .04, \sigma = .2, k = 0.75, g = .02, \alpha = .85$,

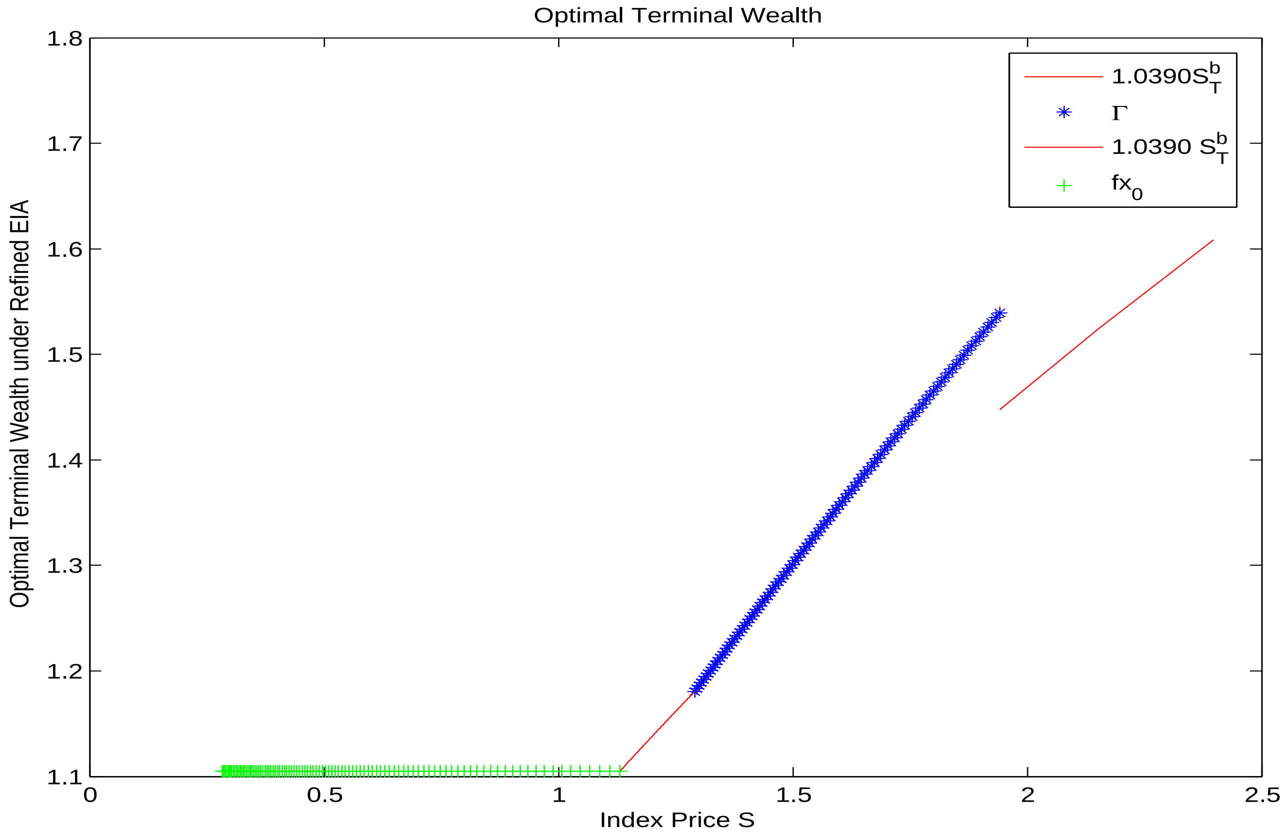

Figure 9: Graph of optimal terminal wealth $k > b$. Green is guaranteed amount $x_0 e^{\gamma T}$. Blue is benchmark $\Gamma = x_0 \left(\frac{S_T}{S_0}\right)^k$ and red is $I(\lambda^* \xi_T)$. Parameters $\mu = .06$, $r = .04$, $\sigma = .2$, $k = 0.75$, $g = .02$, $\alpha = .85$, $\lambda^* = 1.2113$,

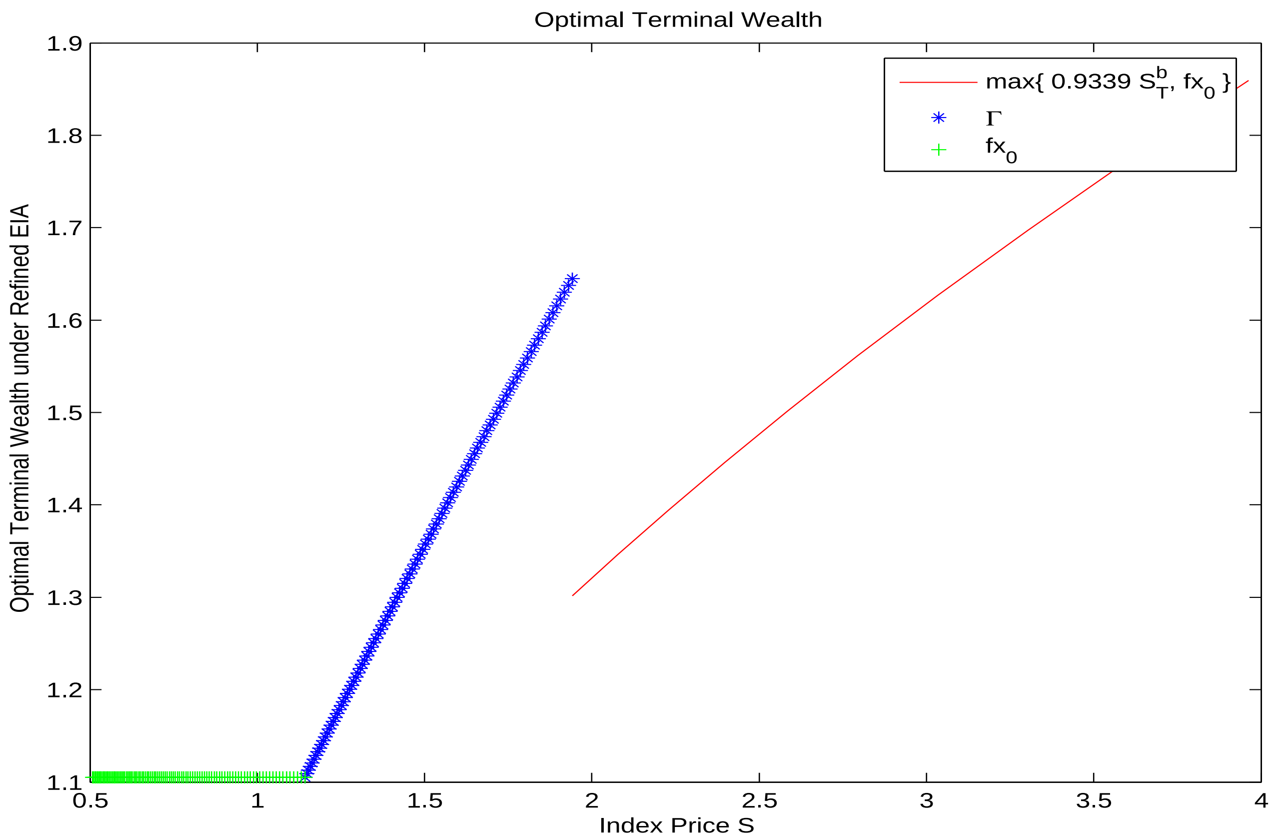

Figure 10: Graph of optimal terminal wealth $k > b$**. Green is guaranteed amount** $x_0 e^{\gamma T}$**. Blue is benchmark** $\Gamma = x_0 \left(\frac{S_T}{S_0}\right)^k$ **and red is** $I(\lambda^* \xi_T)$**. Parameters** $\mu = .06,\ r = .04,\ \sigma = .2,\ k = 0.75,\ g = .02,\ \alpha = .85,\ \lambda^* = 1.2113$ **,**

Tradeoff between parameters

- Can use results from Spivak and Cvitanić (1999) and Cvitanić and Karatzas (1999)
- They solve problem of maximizing the probability that agent's wealth at time T is no less than benchmark.
- Analytical solutions for lognormal case
- This can be used in our context to get trade off between k and α.
- So we can see which contracts are viable

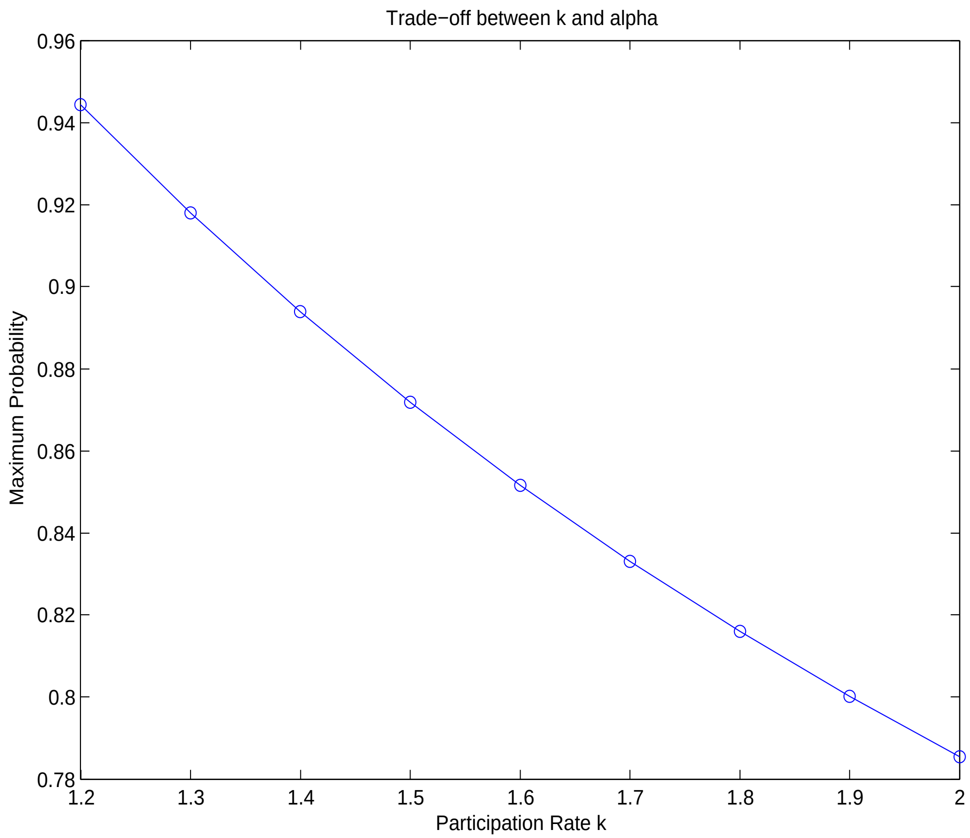

Figure 11: Tradeoff between k and α. $T = 7, \mu = 6\%, r = 4\%, \sigma = 20\%, g = 2\%$.

Discussion

- Graph shows the trade-off between k and the maximum probability α
- Contract parameters: $T = 7, \mu = 6\%, r = 4\%, \sigma = 20\%, g = 2\%$.
- Here $b = 0.5$ and we plot the graph for $k > b$.
- When $k = 2$ the maximum probability $\alpha = 78.55\%$.
- This means there exists no contract with outcome X_T such that
  1. It has guaranteed return at least 2%, and
  2. Beats the index with $k = 2$ with probability higher than 78.55%.
- This means we can readily assess which contracts are viable.

Concerns and questions

- We just considered contract in isolation: in practice there are other assets
- Is this the *right way* to model preferences
- We only considered a static guarantee: we could think about dynamic aspects
- If we write a contract and promise to beat the index with probability α how can it be implemented.
- Contract design based on probability

Summary and future work

- Current EIA's are costly, complex and inefficient.
- We proposed a new contract design
- Maximize expected utility subject to 2 constraints
  1. Minimum return
  2. Beating the benchmark with probability α. This constraint is non convex. We showed existence of solution and gave the functional form of terminal wealth. Proof by construction
- Illustrated with examples.
- Rich patten of behaviour.
- Need to work out more examples and consider replication.

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